

# APPLICATIONS OF THE OHSAWA-TAKEGOSHI EXTENSION THEOREM TO DIRECT IMAGE PROBLEMS

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## Abstract

In the first part of the paper, we study a Fujita-type conjecture by Popa and Schnell, and give an effective bound on the generic global generation of the direct image of the twisted pluricanonical bundle. We also point out the relation between the Seshadri constant and the optimal bound. In the second part, we give an affirmative answer to a question by Demailly-Peternell-Schneider in a more general setting. As an application, we generalize the theorems by Fujino and Gongyo on images of weak Fano manifolds to the Kawamata log terminal cases.

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## 1 Introduction

The first goal of this paper is to study the following conjecture by Popa and Schnell:

**Conjecture 1.1.** (*Popa-Schnell*) *Let  $f : X \rightarrow Y$  be a morphism of smooth projective varieties, with  $\dim(Y) = n$ , and let  $L$  be an ample line bundle on  $Y$ . Then, for every  $k \geq 1$ , the sheaf*

$$f_*(K_X^{\otimes k}) \otimes L^l$$

*is globally generated for any  $l \geq k(n+1)$ .*

In [PS14], Popa and Schnell proved the conjecture in the case when  $L$  is an ample and globally generated line bundle, and in general when  $\dim(X) = 1$ . In a recent preprint [Dut17], Dutta was able to remove the global generation assumption on  $L$  making a statement about generic global generation with weaker bound on the twist, as in the work of Angehrn and Siu [AS95], on the effective freeness of adjoint bundles. Her theorem is as follows:

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**Theorem 1.1.** (Dutta) Let  $f : X \rightarrow Y$  be a morphism of smooth projective varieties, with  $\dim(Y) = n$ , and let  $L$  be an ample line bundle on  $Y$ . Then, for every  $m \geq 1$ , the sheaf

$$f_*(K_X^{\otimes k}) \otimes L^l$$

is generated by global sections at a general point  $y \in Y$ , either

(a) for all  $l \geq k\binom{n+1}{2} + 1$   
or

(b) for all  $l \geq k(n+1)$  when  $n \leq 4$ .

Here  $\binom{n+1}{2}$  is the Angehrn-Siu type bound in their work on the Fujita conjecture [AS95].

Inspired by Demailly's recent work on the Ohsawa-Takegoshi type extension theorem [Dem15] and Păun's proof of Siu's invariance of plurigena [Pau07], we are able to prove the following theorem:

**Theorem A.** Let  $f : X \rightarrow Y$  be a morphism of smooth projective varieties, with  $\dim(Y) = n$ , and let  $L$  be an ample line bundle on  $Y$ . If  $y$  is a regular value of  $f$ , then for every  $k \geq 1$ , the sheaf

$$f_*(K_X^{\otimes k}) \otimes L^l$$

is generated by global sections at  $y$  for any  $l \geq k\left(\left\lfloor \frac{n}{\epsilon(L,y)} \right\rfloor + 1\right)$ . Here  $\epsilon(L, y) > 0$  is the Seshadri constant of  $L$  at the point  $y$ .

Motivated in part by his study of linear series in connection with the Fujita conjecture, Demailly introduced the *Seshadri constant* to measure the local positivity of the ample line bundle at a point [Dem92]. After that Ein and Lazarsfeld systematically studied the Seshadri constant, and they first proved that for any ample line bundle  $L$  on a projective surface  $Y$ , the Seshadri constant

$$\epsilon(L, y) \geq 1$$

for a very general point on  $Y$  [EL93]. Inspired by this result, they further raised the following conjecture:

**Conjecture 1.2.** (Ein-Lazarsfeld) Let  $Y$  be any projective manifold, and  $L$  any ample line bundle on  $Y$ . Then the Seshadri constant

$$\epsilon(L, y) \geq 1$$

at a very general point  $y \in Y$ .

In [EKL95], they proved the existence of universal generic bound in a fixed dimension. However, the bound is suboptimal by a factor of  $n = \dim(Y)$ .

**Theorem 1.2.** (Ein-Küchle-Lazarsfeld) Let  $Y$  be a projective variety, and  $L$  an ample line bundle on  $Y$ . Then for any given  $\delta > 0$ , the locus

$$\{y \in Y \mid \epsilon(L, y) > \frac{1}{n + \delta}\}$$

contains a Zariski-dense open set in  $Y$ .

Applying Theorem 1.2 to our Theorem A, we have the following general result:

**Theorem B.** Let  $f : X \rightarrow Y$  be a morphism of smooth projective varieties, with  $\dim(Y) = n$ , and let  $L$  be an ample line bundle on  $Y$ . Then for any  $k \geq 1$ , the direct image

$$f_*(K_X^{\otimes k}) \otimes L^l$$

is generated by global sections at the generic points of  $Y$  for any  $l \geq k(n^2 + 1)$ . In particular, if the manifold  $Y$  satisfies Conjecture 1.2, then Conjecture 1.1 holds true for general points in  $Y$ ; that is, the direct image

$$f_*(K_X^{\otimes k}) \otimes L^l$$

is generated by global sections at the generic points of  $Y$  for any  $l \geq k(n+1)$ .

Compared to Theorem 1.1 by Dutta, our bound for  $l$  is also quadratic on  $n$  but slightly weaker than hers. However, if we apply the result that  $K_Y + (n+1)L$  is semi-ample for any ample line bundle  $L$ , we can obtain a linear bound for  $l$ .

**Theorem C.** *Let  $f : X \rightarrow Y$  be a morphism of smooth projective varieties, with  $\dim(Y) = n$ , and let  $L$  be an ample line bundle on  $Y$ . Then for every  $k \geq 1$ , the sheaf*

$$f_*(K_X^{\otimes k}) \otimes L^{\otimes l}$$

*is generated by global sections at the generic  $y \in Y$  for any  $l \geq k(n+1) + n^2 - n$ .*

The second part of the paper is to study a question by Demailly-Peternell-Schneider in [DPS01]:

**Problem 1.1.** *Let  $X$  and  $Y$  be normal projective  $\mathbb{Q}$ -Gorenstein varieties. Let  $f : X \rightarrow Y$  be a surjective morphism. If  $-K_X$  is pseudo-effective and its non-nef locus does not project onto  $Y$ , is  $-K_Y$  pseudo-effective?*

Inspired by the recent work of J. Cao on the local isotriviality on the Albanese map of projective manifolds with nef anticanonical bundles [Cao16], we give an affirmative answer to the above problem when  $X$  and  $Y$  are smooth pairs:

**Theorem D.** *Let  $f : X \rightarrow Y$  be a surjective morphism from a log-canonical (lc for short) pair  $(X, D)$  to the smooth projective manifold  $Y$ . Let  $\Delta$  be a (not necessarily effective)  $\mathbb{Q}$ -divisor on  $Y$ . Suppose that  $-(K_X + D) - f^*\Delta$  is pseudo-effective, and the non-nef locus  $\mathbf{B}_-(-(K_X + D) - f^*\Delta)$  does not project onto  $Y$ . Then  $-K_Y - \Delta$  is pseudo-effective with its non-nef locus contained in  $f(\mathbf{B}_-(-(K_X + D) - f^*\Delta)) \cup Z \cup Z_D$ , where  $Z$  is the minimal proper subvariety on  $Y$  such that  $f : X \setminus f^{-1}(Z) \rightarrow Y \setminus Z$  is a smooth fibration, and  $Z_D$  is an at most countable union of proper subvarieties containing  $Z$  such that for every  $y \notin Z_D$ , the pair  $(f^{-1}(y), D|_{f^{-1}(y)})$  is also lc.*

The following theorem by Fujino and Gongyo [FG14] is a direct consequence of our Theorem D.

**Theorem 1.3.** (Fujino-Gongyo) *Let  $f : X \rightarrow Y$  be a smooth fibration between smooth projective varieties. Let  $D$  be an effective  $\mathbb{Q}$ -divisor on  $X$  such that  $(X, D)$  is lc,  $\text{Supp}(D)$  is a simple normal crossing divisor, and  $\text{Supp}(D)$  is relatively normal crossing over  $Y$ . Let  $\Delta$  be a (not necessarily effective)  $\mathbb{Q}$ -divisor on  $Y$ . Assume that  $-(K_X + D) - f^*\Delta$  is nef. Then so is  $-K_Y - \Delta$ .*

Moreover, we can also use analytic methods to prove the following theorem.

**Theorem E.** *Let  $f : X \rightarrow Y$  be a surjective morphism between two smooth manifolds  $X$  and  $Y$ . Let  $D$  be an effective  $\mathbb{Q}$ -divisor such that  $(X, D)$  is klt. Let  $\Delta$  be a (not necessarily effective)  $\mathbb{Q}$ -divisor on  $Y$ . If  $-K_X - D - f^*\Delta$  is big and its non-nef locus  $\mathbf{B}_-(-K_X - D - f^*\Delta)$  does not project onto  $Y$ , then  $-K_Y - \Delta$  is big.*

As a combination of Theorem D and E, we prove the following Theorem, which is a generalization of a theorem by Fujino and Gongyo [FG12].

**Theorem F.** *Let  $f : X \rightarrow Y$  be a smooth fibration between two smooth manifolds  $X$  and  $Y$ . Let  $D$  be an effective  $\mathbb{Q}$ -divisor such that  $(X, D)$  is klt, and  $(X_y, D|_{X_y})$  is also klt for every  $y \in Y$ . Let  $\Delta$  be a (not necessarily effective)  $\mathbb{Q}$ -divisor on  $Y$ . If  $-K_X - D - f^*\Delta$  is big and nef, then  $-K_Y - \Delta$  is also big and nef.*

## 2 PRELIMINARY TECHNIQUES

### 2.1 SESHADRI CONSTANTS

In the work [Dem92], Demailly define the following *Seshadri constant*:

**Definition 2.1.** Let  $L$  be a nef line bundle over a projective algebraic manifold  $X$ . To every point  $x \in X$ , one defines the number

$$\epsilon(L, x) := \inf \frac{L \cdot C}{\nu(C, x)}$$

where the infimum is taken over all reduced irreducible curves  $C$  passing through  $x$  and  $\nu(C, x)$  is the multiplicity of  $C$  at  $x$ .  $\epsilon(L, x)$  will be called the Seshadri constant  $L$  at  $x$ .

On the other hand, Demailly also introduced another constant  $\gamma(L, x)$  for any nef line bundle  $L$ . First, we begin with the following definition.

**Definition 2.2.** A function  $\psi : X \rightarrow ]-\infty, +\infty]$  on a complex manifold  $X$  of dimension  $m$  is said to be quasi-plurisubharmonic (quasi-psh for short) if  $\psi$  is locally the sum of a psh function and of a smooth function (or equivalently, if  $\sqrt{-1}\partial\bar{\partial}\psi$  is locally bounded from below). In addition, we say that  $\psi$  has neat analytic singularities if every point  $x \in X$  possesses an open neighborhood  $U$  on which  $\psi$  can be written

$$\psi = c \log \sum_{j=1}^N |g_j|^2 + w(z)$$

where  $g_j \in \mathcal{O}(U)$ ,  $c \geq 0$  and  $w(z) \in \mathcal{C}^\infty(U)$ .

**Definition 2.3.** A singular metric  $h$  on the line bundle  $L$  is said to have a logarithmic pole of coefficient  $\nu$  at a point  $x \in X$ , if on a neighborhood  $U$  of  $x$ , the local weight  $\varphi$  of  $h$  can be written

$$\varphi = \nu \log \sum |z - x|^2 + w(z)$$

where  $\nu > 0$  and  $w(z) \in \mathcal{C}^\infty(U)$ . In this setting, we set  $\nu(h, x) := \nu$ .

Then we set

$$\gamma(L, x) := \sup_h \nu(h, x),$$

where the supremum is taken over all singular hermitian metrics  $h$  of  $L$  with positive curvature current, whose local weight  $\varphi$  has neat singularities and logarithmic poles at  $x$ .

The numbers  $\epsilon(L, x)$  and  $\gamma(L, x)$  will be seen to carry a lot of useful information about the local positivity of  $L$ . In case  $L$  is big and nef, these two constants coincide outside a certain proper subvariety of  $X$  (see [Dem92, Theorem 6.4])

**Theorem 2.1.** (Demailly) Let  $L$  be a big and nef line bundle over  $X$ . Then we have

$$\epsilon(L, x) = \gamma(L, x)$$

for any  $x \notin \mathbf{B}_+(L)$ , where  $\mathbf{B}_+(L)$  is the augmented base locus of  $L$  (see [Laz04, Definition 10.2.2]). In particular, if  $L$  is ample, then  $\epsilon(L, x) = \gamma(L, x)$  holds everywhere.

As we mentioned in Section 1, in [EKL95], Ein, Küchle and Lazarsfeld gave the existence of universal generic bounds for the Seshadri constants in a fixed dimension.

**Theorem 2.2.** (Ein-Küchle-Lazarsfeld) Let  $Y$  be an irreducible projective variety of dimension  $n$ , and  $L$  a nef line bundle on  $Y$ . Suppose there exists a countable union  $\mathcal{B} \subset Y$  of proper subvarieties of  $Y$  plus a positive real number  $\alpha > 0$  such that

$$L^r \cdot Z \geq (\alpha \cdot r)^r \tag{2.1}$$

for every irreducible subvariety  $Z \subset Y$  of dimension  $r$  ( $1 \leq r \leq n$ ) with  $Z \not\subset \mathcal{B}$ . Then

$$\epsilon(L, y) \geq \alpha$$

for all  $y \in Y$  outside a countable union of proper subvarieties in  $Y$ . In particular, for any ample line bundle  $L$  on  $Y$ ,

$$\epsilon(L, y) \geq \frac{1}{n} \tag{2.2}$$

for a very general point  $y$ .

The above theorem gives a lower bound on the Seshadri constant of a nef and big line bundle at a very general point. However, as was also proved in [EKL95], for the ample line bundle, the above theorem is valid on a Zariski-open set by the semi-continuity of the Seshadri constant of the ample line bundle. In other word, let  $L$  be an ample line bundle on an irreducible projective variety  $Y$ . Suppose that there is a positive rational number  $B$  and a smooth point  $y \in Y$  for which one knows that

$$\epsilon(L, y) > B.$$

Then the locus

$$\{z \in Y | \epsilon(L, z) > B\}$$

contains a Zariski-open dense set in  $Y$ .

## 2.2 $L^2$ EXTENSION THEOREM

Before we state Demailly's Ohsawa-Takegoshi type Extension Theorem, we begin with a definition in [Dem15].

**Definition 2.4.** *If  $\psi$  is a quasi-psh function on a complex manifold  $X$ , the multiplier ideal sheaf  $\mathcal{J}(\psi)$  is the coherent analytic subsheaf of  $\mathcal{O}_X$  defined by*

$$\mathcal{J}(\psi)_x := \{f \in \mathcal{O}_{X,x}; \exists U \ni x, \int_U |f|^2 e^{-\psi} d\lambda < +\infty\}$$

where  $U$  is an open coordinate neighborhood of  $x$ , and  $d\lambda$  the standard Lebesgue measure in the corresponding open chart of  $\mathbb{C}^n$ . We say that the singularities of  $\psi$  are log canonical along the zero variety  $Y := V(\mathcal{J}(\psi))$  if  $\mathcal{J}((1-\epsilon)\psi)|_Y = \mathcal{O}_{X|Y}$  for every  $\epsilon > 0$ .

If  $\psi$  possesses both neat and log canonical singularities, it is easy to show that the zero scheme  $V(\mathcal{J}(Y))$  is a reduced variety. In this case one can also associate in a natural way a measure  $dV_{Y^\circ, \omega}[\psi]$  on the set  $Y^\circ := Y^{\text{reg}}$  of regular points of  $Y$  as follows. If  $g \in \mathcal{C}_c(Y^\circ)$  is a compactly supported continuous function on  $Y^\circ$ , and  $\tilde{g}$  compactly supported extension of  $g$  to  $X$ , we set

$$\int_{Y^\circ} g dV_{Y^\circ, \omega}[\psi] := \limsup_{t \rightarrow -\infty} \int_{x \in X, t < \psi(x) < t+1} \tilde{g}(x) dV_{X, \omega}. \quad (2.3)$$

Here  $\omega$  is a Kähler metric on  $X$ , and  $dV_{X, \omega} = \frac{\omega^m}{m!}$ . In [Dem15] Demailly proved that the limit does not depend on the continuous extension  $\tilde{g}$ , and one gets in this way a measure with smooth positive density with respect to the Lebesgue measure, at least on an (analytic) Zariski open set in  $Y^\circ$ .

We are ready to recall the Ohsawa-Takegoshi type extension Theorem by Demailly. We only need a special case of his very general statement:

**Theorem 2.3.** *(Demailly) Let  $X$  be a smooth projective manifold, and  $\omega$  a Kähler metric on  $X$ . Let  $L$  be a holomorphic line bundle equipped with a (singular) hermitian metric  $h$  on  $X$ , and let  $\psi : X \rightarrow ]-\infty, +\infty]$  be a quasi-psh function on  $X$  with neat analytic singularities. Let  $Y$  be the analytic subvariety of  $X$  defined by  $Y = V(\mathcal{J}(Y))$  and assume that  $\psi$  has log canonical singularities along  $Y$ , so that  $Y$  is reduced. Finally, assume that the curvature current*

$$i\Theta_{L, h} + \alpha \sqrt{-1} \partial \bar{\partial} \psi \geq 0$$

for all  $\alpha \in [1, 1 + \delta]$  and some  $\delta > 0$ . Then for every section  $s \in H^0(Y^\circ, (K_X \otimes L)|_{Y^\circ})$  on  $Y^\circ := Y^{\text{reg}}$  such that

$$\int_{Y^\circ} |s|_{\omega, h}^2 dV_{Y^\circ, \omega}[\psi] < +\infty,$$

there is an extension of  $S \in H^0(X, K_X \otimes L)$  whose restriction to  $Y^\circ$  is equal to  $s$ , such that

$$\int_X \gamma(\delta \psi) |S|_{\omega, h}^2 e^{-\psi} dV_{X, \omega} \leq \frac{34}{\delta} \int_{Y^\circ} |s|_{\omega, h}^2 dV_{Y^\circ, \omega}[\psi].$$

Here we set

$$\gamma = \begin{cases} e^{-\frac{x}{2}} & \text{if } x \geq 0, \\ \frac{1}{1+x^2} & \text{if } x < 0. \end{cases}$$

A direct consequence of Theorem 2.3 is the following extension theorem for fibrations:

**Corollary 2.1.** *Let  $f : X \rightarrow Y$  be a surjective morphism between smooth manifolds. For any ample line bundle  $L$  on  $Y$ , any regular value  $y$  of  $f$ , if the Seshadri constant of  $L$  satisfies that*

$$\epsilon(L, y) > \dim(Y) = n, \quad (2.4)$$

*then for any pseudo-effective line bundle  $L_1$  over  $X$  with a singular hermitian metric  $h$  such that  $\Theta_{L_1, h} \geq 0$ , and the restriction of  $h$  to  $X_y$  is not identically zero, any section  $s$  of*

$$H^0(X_y, (K_X \otimes f^*L \otimes L_1)|_{X_y} \otimes \mathcal{J}(h|_{X_y})).$$

*can always be extended to a global one*

$$S \in H^0(X, K_X \otimes f^*L \otimes L_1)$$

*with certain  $L^2$  estimates which do not depend on  $L_1$ .*

*Proof.* Since  $L$  is ample over  $Y$ , one can find a smooth hermitian metric  $h_0$  on  $L$  with the curvature form  $i\Theta_{L, h_0} \geq \omega$ , where  $\omega$  is some Kähler form on  $Y$ .

By the lower bound of Seshadri constant  $\epsilon(L, y) > n$ , we can find a global quasi-psh function  $\varphi$  with neat singularities on  $Y$  such that

- (a)  $i\Theta_{L, h_0} + \sqrt{-1}\partial\bar{\partial}\varphi \geq 0$ ;
- (b)  $\varphi$  is smooth outside  $y$ ;
- (c) on a neighborhood  $W$  of  $y$ , we have

$$\varphi = (1 + \delta)n \log \sum |z - y|^2 + w(z)$$

where  $\delta > 0$  and  $w(z) \in \mathcal{C}^\infty(W)$  with  $w(y) = 0$

Now set  $\psi := \frac{1}{1+\delta}\varphi \circ f$ , which is a quasi-psh function with neat singularities on  $X$ . Moreover, since  $y$  is the regular value of  $f$ , the inverse image  $X_y := f^{-1}\{y\}$  is a closed smooth submanifold of codimension  $n$  in  $X$ , and the multiplier ideal sheaf

$$\mathcal{J}(\psi) = \mathcal{J}(\mathcal{I}_{X_y}^{(n)}) = \mathcal{I}_{X_y}.$$

Here  $\mathcal{I}_{X_y}^{(n)}$  is the ideal sheaf consisting of germs of functions that have multiplicity  $\geq n$  at a general point of  $X_y$ :

$$\mathcal{I}_{X_y}^{(n)} := \{f \in \mathcal{O}_X \mid \text{ord}_x(f) \geq n \text{ for a general point } x \in X\}.$$

Thus  $\mathcal{J}(\psi)$  has log canonical singularities, and we have

$$i\Theta_{L_1, h} + i\Theta_{f^*L, f^*h_0} + \alpha\sqrt{-1}\partial\bar{\partial}\psi \geq 0$$

for all  $\alpha \in [1, 1 + \delta]$ . Then for any section  $s$  of

$$H^0(X_y, (K_X \otimes f^*L \otimes L_1)|_{X_y} \otimes \mathcal{J}(h|_{X_y})),$$

we can apply Theorem 2.3 to extend  $s$  to a global section

$$S \in H^0(X, K_X \otimes f^*L \otimes L_1 \otimes \mathcal{J}(h))$$

such that

$$\int_X \gamma(\delta\psi) |S|_{\omega, f^* h_0 h_1}^2 e^{-\psi} dV_{X, \omega} \leq \frac{34}{\delta} \int_{X_y} |s|_{\omega, f^* h_0 h_1}^2 dV_{X_y, \omega}[\psi].$$

Assume that  $\dim(X) = m + n$ . From (2.3) one can then check that  $dV_{X_y, \omega}[\psi]$  is the smooth measure supported on  $X_y$ , such that

$$dV_{X_y, \omega}[\psi] = C_0 \frac{\omega_{\upharpoonright X_y}^m}{m!},$$

where  $C_0$  is some constant depending only on  $m, n$ . Since  $\delta$  depends only on  $\epsilon(L, y)$ , write  $C := \frac{34}{\delta} C_0$  which does not depend on  $L_1$ . We thus obtain

$$\int_X \gamma(\delta\psi) |S|_{\omega, f^* h_0 h_1}^2 e^{-\psi} dV_{X, \omega} \leq C_0 \int_{X_y} |s|_{\omega, f^* h_0 h_1}^2 \frac{\omega_{\upharpoonright X_y}^m}{m!}, \quad (2.5)$$

where the  $L^2$  estimate does not depend on  $L_1$ .  $\square$

### 2.3 THE EXTENSION THEOREM FOR TWISTED PLURICANONICAL BUNDLES

We recall the following twisted pluricanonical extension theorem, which was recently used by J. Cao to prove the local triviality of Albanese maps of projective manifolds with nef anticanonical bundles [Cao16]. It is a consequence of [BP10, Section A.2].

**Theorem 2.4.** *Let  $Y$  be a  $n$ -dimensional projective manifold and let  $A_Y$  be any line bundle on  $Y$  such that the difference  $A_Y - K_Y$  is an ample line bundle. Let  $f : X \rightarrow Y$  be a surjective morphism from a smooth projective manifold  $X$  to  $Y$  and  $L$  be a pseudo-effective line bundle on  $X$  with a possible singular metric  $h_L$  such that*

$$i\Theta_{h_L}(L) \geq 0.$$

*Then for  $y \in Y$  such that*

- (a)  *$y$  is the regular value of  $f$ ,*
- (b) *the Seshadri constant  $\epsilon(A_Y - K_Y, y) > n$ ,*
- (c)  *$\mathcal{J}(h_{L \upharpoonright X_y}^{\frac{1}{m}}) = \mathcal{O}_{X_y}$ ,*

*the restriction map*

$$H^0(X, mK_{X/Y} + L + f^* A_Y) \rightarrow H^0(X_y, mK_{X_y} + L_{\upharpoonright X_y})$$

*is surjective. In particular, the choice of  $A_Y$  depends only on  $Y$  and is independent of  $f, L, m$ .*

*Proof.* Thanks to [BP10, A.2.1], if for some regular value  $y$ , we have

$$\mathcal{J}(h_{L \upharpoonright X_y}^{\frac{1}{m}}) = \mathcal{O}_{X_y} \quad (2.6)$$

and

$$H^0(X_y, mK_{X_y} + L_{\upharpoonright X_y}) \neq \emptyset,$$

there exists a  $m$ -relative Bergman type metric  $h_{m, B}$  on  $mK_{X/Y} + L$  with respect to  $h_L$  such that  $i\Theta_{h_{m, B}}(mK_{X/Y} + L) \geq 0$ . Thus  $h := \frac{m-1}{m} h_{m, B} + \frac{1}{m} h_L$  defines a possible singular metric on

$$\tilde{L} := \frac{m-1}{m} (mK_{X/Y} + L) + \frac{1}{m} L = (m-1)K_{X/Y} + L,$$

with  $i\Theta_h(\tilde{L}) \geq 0$ .



Take any  $s \in H^0(X_y, mK_{X/Y} + L)$ , by the construction of the  $m$ -relative Bergman kernel metric,  $|s|_{h_{m,B}}^2$  is  $\mathcal{C}^0$ -bounded. Combining this with the klt assumption (2.6), we see that

$$\begin{aligned} \int_{X_y} |s|_{\omega,h}^2 dV_{X_y,\omega} &= \int_{X_y} |s|_{h_{m,B}}^{\frac{2(m-1)}{m}} |s|_{\omega,h_{\frac{1}{L}}^{\frac{2}{m}}}^{\frac{2}{m}} dV_{X_y,\omega} \\ &\leq C \int_{X_y} |s|_{\omega,h_{\frac{1}{L}}^{\frac{2}{m}}}^{\frac{2}{m}} dV_{X_y,\omega} < +\infty. \end{aligned}$$

We then can apply Corollary 2.1 to  $K_X + \tilde{L} + f^*(A_Y - K_Y)$ , to extend  $s$  to a section in  $H^0(X, K_{X/Y} + \tilde{L} + f^*A_Y)$ . In conclusion, the restriction

$$H^0(X, mK_{X/Y} + L + f^*A_Y) \rightarrow H^0(X_y, mK_{X_y} + L|_{X_y})$$

is surjective and the theorem is proved.  $\square$

### 3 ON THE CONJECTURE OF POPA AND SCHNELL

Let  $f : X \rightarrow Y$  be the surjective morphism between smooth projective manifolds, and let  $L$  be an ample line bundle on  $Y$  with a smooth hermitian metric  $h_0$  such that the curvature form  $i\Theta_{h_0} \geq \omega$  for some Kähler metric  $\omega$  on  $Y$ . Assume that  $\dim(Y) = n$  and  $\dim(X) = m + n$ . Fix any point  $y$  on  $Y$  which is the regular value of  $f$ . Take any positive real number  $\nu$  such that

$$\epsilon(L, y) > \frac{1}{\nu}.$$

Then we have

$$\epsilon([n\nu]L, y) > n.$$

Set  $\tilde{L} := [n\nu]f^*L$  with the smooth hermitian metric  $\tilde{h} := f^*h_0^{[n\nu]}$ , then we can restate Corollary 2.1 in the following variant form:

**Proposition 3.1.** *There is a globally defined quasi-psh function  $\psi_0$  defined over  $X$  and a positive number  $\delta$  such that, for any pseudo-effective line bundle  $L_1$  equipped with the possible singular hermitian metric  $h_1$ , whose curvature current  $i\Theta_{L_1, h_1} \geq 0$  and  $h_1$  is not identically zero when restricted on  $X_y$ , for any section*

$$s \in H^0(X_y, (K_X \otimes \tilde{L} \otimes L_1)|_{X_y} \otimes \mathcal{J}(h_1|_{X_y})),$$

there is a global section

$$S \in H^0(X, K_X \otimes \tilde{L} \otimes L_1)$$

whose restriction to  $X_y$  is  $s$ , such that

$$\int_X \gamma(\delta\psi_0) |S|_{\omega, \tilde{h}h_1}^2 e^{-\psi_0} dV_{X,\omega} \leq C \int_{X_y} |s|_{\omega, \tilde{h}h_1}^2 dV_{X_y,\omega}.$$

Here  $dV_{X_y,\omega} := \frac{\omega|_{X_y}^m}{m!}$ , and  $C$  is some constant which does not depend on  $L_1$ .

Thus from Proposition 3.1, if we set  $L_1$  to be the trivial bundle on  $X$ , we see that the following morphism

$$H^0(X, K_X \otimes f^*L^{\otimes [n\nu]}) \rightarrow H^0(X_y, (K_X \otimes f^*L^{\otimes [n\nu]})|_{X_y})$$

is always surjective. As one can take  $\nu$  to be arbitrary close to  $\frac{1}{\epsilon(L,y)}$  so that  $[n\nu] = \left\lfloor \frac{n}{\epsilon(L,y)} \right\rfloor + 1$ , we see

that the direct image  $f_*K_X \otimes L^{\otimes \left\lfloor \frac{n}{\epsilon(L,y)} \right\rfloor + 1}$  is generated by global sections at  $y$ . Since  $y$  is an arbitrary regular value of  $f$ , we thus prove Theorem A for  $k = 1$ . In order to prove the theorem for any  $k \geq 2$ , we need to apply the techniques in proving Siu's invariance of plurigeners [Siu97] by Păun [Pau07].



*Proof.* (Proof of Theorem A) Fix any  $k \geq 2$  and any  $\sigma \in H^0(X_y, k(K_X + \tilde{L})|_{X_y})$ . We want to find a global section  $\Sigma \in H^0(X, k(K_X + \tilde{L}))$  whose restriction to  $X_y$  is  $\sigma$ .

Choose a very ample line bundle  $A$  on  $X$  such that for every  $r = 0, \dots, k-1$ , the line bundle  $F_{0,r} := r(K_X + \tilde{L}) + A$  is globally generated by sections

$$\{u_j^{(0,r)}\}_{j=1,\dots,N_r} \subset H^0(X, F_{0,r}).$$

We then define inductively a sequence of line bundles

$$F_{q,r} := (qk + r)(K_X + \tilde{L}) + A$$

for any  $q \geq 0$ , and  $0 \leq r \leq k-1$ . By constructions we have

$$\begin{cases} F_{q,r+1} = K_X + F_{q,r} + \tilde{L} & \text{if } r < k-1, \\ F_{q+1,0} = K_X + F_{q,k-1} + \tilde{L} & \text{if } r = k-1. \end{cases} \quad (3.1)$$

We are going to construct inductively families of sections, say  $\{u_j^{(q,r)}\}_{j=1,\dots,N_r}$ , of  $F_{q,r}$  over  $X$ , together with ad hoc  $L^2$  estimates, such that each  $u_j^{(q,r)}$  is an extension of  $v_j^{(q,r)}$ , where we set

$$v_j^{(q,r)} := \sigma^q u_j^{(0,r)}|_{X_y} \in H^0(X, F_{q,r}).$$

Now, by induction, assume that such  $\{u_j^{(q,r)}\}_{j=1,\dots,N_r}$  above can be constructed. Then  $F_{p,r}$  can be equipped with a natural singular hermitian metric  $h_{q,r}$  defined by

$$|\xi|_{h_{q,r}}^2 := \frac{|\xi|^2}{\sum_{j=1}^{N_r} |u_j^{(q,r)}|^2},$$

such that  $i\Theta_{h_{q,r}} \geq 0$ . Let  $h_{K_X}$  be the smooth hermitian metric of the canonical bundle  $K_X$  induced by the volume form  $dV_{X,\omega}$ , and set  $\hat{h} := h_{K_X} \tilde{h}$  to be the smooth metric on  $K_X + \tilde{L}$ , then by construction the pointwise norm with respect to the metric  $h_{q,r}$  is

$$\begin{cases} |v_j^{(q,r+1)}|_{\omega, h_{q,r} \tilde{h}}^2 = \frac{|v_j^{(0,r+1)}|_{\hat{h}^{r+1} h_A}^2}{\sum_{i=1}^{N_r} |v_i^{(0,r)}|_{\hat{h}^r h_A}^2} & \text{if } r < k-1, \\ |v_j^{(q+1,0)}|_{\omega, h_{q,r} \tilde{h}}^2 = \frac{|\sigma|_{\hat{h}^k} |v_j^{(0,0)}|_{h_A}^2}{\sum_{i=1}^{N_r} |v_i^{(0,r)}|_{\hat{h}^{k-1} h_A}^2} & \text{if } r = k-1. \end{cases} \quad (3.2)$$

where  $h_A$  is a smooth hermitian metric on  $A$  with strictly positive curvature. Since the sections  $\{v_i^{(0,r)}\}_{i=1,N_r}$  generates  $F_{0,r}|_{X_y}$ , there is a constant  $C_1 > 0$  such that (3.2) is uniformly  $\mathcal{C}^0$  bounded above by  $C_1$ . From (3.1), it then follows from Proposition 3.1 that one can extend  $v_j^{(q,r+1)}$  (or  $v_j^{(q+1,0)}$  if  $r = k-1$ ) into a section  $u_j^{(q,r+1)}$  ( $u_j^{(q+1,0)}$  respectively) over  $X$  such that

$$\begin{cases} \int_X \gamma(\delta\psi_0) e^{-\psi_0} \sum_{j=1}^{N_{r+1}} |u_j^{(q,r+1)}|_{\omega, h_{q,r} \tilde{h}}^2 dV_{X,\omega} \leq C_2 & \text{if } r < k-1, \\ \int_X \gamma(\delta\psi_0) e^{-\psi_0} \sum_{j=1}^{N_0} |u_j^{(q+1,0)}|_{\omega, h_{q,r} \tilde{h}}^2 dV_{X,\omega} \leq C_2 & \text{if } r = k-1. \end{cases} \quad (3.3)$$

for some uniform constant  $C_2$ . From (3.2), (3.3) is equivalent to

$$\begin{cases} \int_X \gamma(\delta\psi_0) e^{-\psi_0} \frac{\sum_{i=1}^{N_{r+1}} |u_i^{(q,r+1)}|_{\hat{h}^{qk+r+1} h_A}^2}{\sum_{i=1}^{N_r} |u_i^{(q,r)}|_{\hat{h}^{qk+r} h_A}^2} dV_{X,\omega} \leq C_2 & \text{if } r < k-1, \\ \int_X \gamma(\delta\psi_0) e^{-\psi_0} \frac{\sum_{i=1}^{N_0} |u_i^{(q,r+1)}|_{\hat{h}^{qk+k} h_A}^2}{\sum_{i=1}^{N_r} |u_i^{(q,r)}|_{\hat{h}^{qk+k-1} h_A}^2} dV_{X,\omega} \leq C_2 & \text{if } r = k-1. \end{cases} \quad (3.4)$$

Let us denote by

$$a_{qk+r}(x) := \sum_{i=1}^{N_r} |u_i^{(q,r)}|_{\hat{h}^{qk+r}h_A},$$

which is a quasi-psh and bounded non-negative smooth function on  $X$ . By the integrability of  $\log \gamma(\delta\psi_0)$  and  $\psi_0$  with respect to the standard Lebesgue measure over  $X$ , combined with the concavity property of the logarithmic function as well as the Jensen inequality, we can find some constant  $C_3$  and  $C_4$  such that

$$\int_X \log \frac{a_l}{a_{l-1}} dV_{X,\omega} \leq C_3 - \int_X \log \gamma(\delta\psi_0) dV_{X,\omega} + \int_X \psi_0 dV_{X,\omega} \leq C_4 \quad (3.5)$$

for any  $l \geq 1$ . Since  $a_1(x)$  is a bounded smooth function on  $X$ , we can also find a constant  $C_5 \geq C_4$  such that

$$\int_X \log a_1 dV_{X,\omega} \leq C_5.$$

Combined these inequalities together we obtain

$$\int_X \frac{\log a_l}{l} dV_{X,\omega} \leq C_5$$

for any  $l \geq 1$ . Set  $f_q := \frac{\log a_{qk}}{q}$ , and we have the following properties:

(a) for any  $q \geq 1$ , we have

$$\int_X f_q dV_{X,\omega} \leq C_5;$$

(b) the inequality

$$k\Theta_{\hat{h}}(K_X + \tilde{L}) + \sqrt{-1}\partial\bar{\partial}f_q \geq -\frac{1}{q}\Theta_{h_A}(A)$$

holds true in the sense of currents on  $X$ ;

(c) on  $X_y$  the following equality is satisfied

$$f_q|_{X_y} = \log |\sigma|_{\hat{h}^k}^2 + a_0(x)|_{X_y}$$

where  $a_0(x) = \log \sum_{i=1}^{N_0} |u_i^{(0,0)}|_{h_A}$  is a smooth function on  $X$ .

By the mean value inequality for the psh functions, as a consequence of the properties (a) and (b), one can show the existence of a uniform upper bound for the functions  $f_q$  over  $X$ . Thus the sequence  $f_q(z)$  must have some subsequence which converges in  $L^1$  topology on  $X$  to the potential  $f_\infty$ , in the form of the regularized limit

$$f_\infty(z) := \limsup_{\zeta \rightarrow z} \lim_{q_\nu \rightarrow +\infty} f_{q_\nu}(\zeta),$$

which satisfies

$$k\Theta_{\hat{h}}(K_X + \tilde{L}) + \sqrt{-1}\partial\bar{\partial}f_\infty \geq 0$$

as a current on  $X$ . Moreover, by Property (c)  $f_\infty$  is not identically  $-\infty$  on  $X_y$ , as well as

$$f_\infty \geq \log |\sigma|_{\hat{h}^k}^2 + \mathcal{O}(1) \quad (3.6)$$

pointwise on  $X_y$ .

Now we construct a singular hermitian metric  $h_\infty$  on  $(k-1)(K_X + L)$  defined by

$$h_\infty := \hat{h}^{k-1} e^{-\frac{k-1}{k}f_\infty}.$$

Then  $\Theta_{h_\infty}((k-1)(K_X + \tilde{L})) \geq 0$ . Write  $k(K_X + L) = K_X + (k-1)(K_X + \tilde{L}) + \tilde{L}$ , where  $(k-1)(K_X + \tilde{L})$  is equipped with the singular hermitian metric  $h_\infty$ . Since

$$|\sigma|_{\omega, \tilde{h}h_\infty}^2 = |\sigma|_{\tilde{h}h_\infty}^2 = |\sigma|_{h_\infty}^{\frac{2(k-1)}{k}} \cdot |\sigma|_{\tilde{h}}^{\frac{2}{k}}$$

which is  $\mathcal{C}^0$  bounded, we then can apply Proposition 3.1 to extend  $\sigma$  to a global section  $\Sigma \in H^0(X, k(K_X + \tilde{L}))$ .

In conclusion, for any regular value  $y$  of the morphism  $f$ , the following morphism

$$H^0(X, K_X^{\otimes k} \otimes f^* L^{\otimes l}) \rightarrow H^0(X_y, (K_X^{\otimes k} \otimes f^* L^{\otimes l})|_{X_y})$$

is always surjective for any  $l > \frac{n}{\epsilon(L, y)}$ . Thus Theorem A is proved.  $\square$

In order to improve the above quadratic bound to linear, we need to apply the twisted pluri-canonical extension theorem in Section 2.3 instead. First, we recall the following result arising from birational geometry:

**Theorem 3.1.** *Let  $L$  be an ample line bundle over a projective  $n$ -fold  $Y$ , then the adjoint line bundle  $K_Y + (n + 1)L$  is semi-ample.*

Based on the Mori theory, one observes that  $n + 1$  is the maximal length of extremal rays of smooth projective  $n$ -folds, which shows that  $K_Y + (n + 1)L$  is nef. By the base-point-free theorem, one can even show that  $K_Y + (n + 1)L$  is semiample. In his work on the Fujita conjecture [Dem96], Demailly also gave an analytic proof for the fact that  $K_Y + (n + 1)L$  is nef.

*Proof.* (Proof of Theorem C) By Theorem 2.2, the Seshadri constant

$$\epsilon((n^2 + 1)L, y) > n$$

for a generic  $y \in Y$ . From Theorem 3.1, we see that  $K_Y + (n + 1)L$  can be equipped with a smooth hermitian metric  $h$  with semi-positive curvature. Applying Theorem 2.4, we see that for any  $m \geq 1$ , the restriction map

$$H^0\left(X, mK_{X/Y} + (m - 1)f^*(K_Y + (n + 1)L) + f^*(K_Y + (n^2 + 1)L)\right) \rightarrow H^0(X_y, mK_{X|X_y})$$

is surjective for a generic  $y$  in  $Y$ . In other words, for any  $k \geq 1$  and any  $l \geq k(n + 1) + n^2 - n$ , the direct image

$$f_*(K_X^{\otimes k}) \otimes L^{\otimes l}$$

is generated by global sections at the generic points of  $Y$ . This completes the proof of Theorem C.  $\square$

## 4 ON A QUESTION OF DEMAILLY-PETERNELL-SCHNEIDER

In this section, we prove Theorem D and thus give an affirmative answer to Problem 1.1 in the case that both  $X$  and  $Y$  are smooth manifolds.

*Proof.* (Proof of Theorem D) Take a sufficient ample line bundle  $A_X$  on  $X$  such that  $A_X + D$  is ample, and the direct image  $f_*(A_X)$  is a torsion free coherent sheaf which is not only locally free but also globally generated over the Zariski open set  $X^\circ := X \setminus f^{-1}(Z)$ . Then  $f_*(A_X)$  is locally free outside a subvariety  $W \subset Z$  of codimension at least 2. Set  $r$  to be the generic rank of  $f_*(A_X)$ , and denote by

$$\det f_*(A_X) := \wedge^r f_*(A_X)^{**}$$

to be the bidual of  $\wedge^r f_*(A_X)$  which is an invertible sheaf over  $Y$ , then there is coherent ideal sheaf  $\mathcal{I}$  supported on  $W$  such that

$$\wedge^r f_*(A_X) = \det f_*(A_X) \otimes \mathcal{I}.$$

Take a smooth hermitian metric  $h$  on  $A_X + D$  such that  $i\Theta_h \geq 3\omega$  for some Kähler metric  $\omega$ . Let us also choose a very ample line bundle  $A_Y$  on  $Y$  such that  $A_Y - K_Y$  generates  $n + 1$  jets everywhere and  $A_Y + \det f_*(A_X)$  is also an ample line bundle on  $Y$ . In particular, the Seshadri constant  $\epsilon(A_Y - K_Y, y) > n$  for any  $y$ .

By the definition of non-nef locus, for any pseudo-effective line bundle  $E$  on  $X$ , we have

$$\mathbf{B}_-(E) = \bigcup_{m \in \mathbb{N}} \bigcap_{k \in \mathbb{N}} \text{Bs}(kA_X + kmE).$$

Equivalently, in [BDPP13], it is shown that

$$\mathbf{B}_-(E) = \bigcup_{m \in \mathbb{N}} \bigcap_T E_+(T),$$

where  $T$  runs over the set  $c_1(E)[- \frac{1}{m}\omega]$  of all closed real  $(1,1)$ -currents  $T \in c_1(E)$  such that  $T \geq -\frac{1}{m}\omega$ , and  $E_+(T)$  denotes the locus where the Lelong numbers of  $T$  are strictly positive. By [Bou02], there is always a current  $T_{\min, m}$  which achieves minimum singularities and minimum Lelong numbers among all members of  $c_1(E)[- \frac{1}{m}\omega]$  hence

$$\mathbf{B}_-(E) = \bigcup_{m \in \mathbb{N}} E_+(T_{\min, m}).$$

By Demailly's regularization theorem in [Dem92b], for every  $m \in \mathbb{N}$ , we can find a closed  $(1,1)$ -current  $T_m \in c_1(E)$  with neat singularities such that  $T_m \geq -\frac{2}{m}\omega$ , and

$$E_+(T_{\min, 2m}) \subset E_+(T_m) \subset E_+(T_{\min, m}).$$

Equivalently, there exists a singular hermitian metric  $\tilde{h}_m$  on  $E$  with neat singularities, such that the curvature current

$$i\Theta_{\tilde{h}_m} = T_m \geq -\frac{2}{m}\omega.$$

Set  $E := -(K_X + D) - f^*\Delta$ . Since  $\mathbf{B}_-(-(K_X + D) - f^*\Delta)$  does not project onto  $Y$ , thus for any  $m \in \mathbb{N}$ ,  $Z_m := f(E_+(T_m))$  is a proper subvariety of  $Y$ , and the singular hermitian metric  $\tilde{h}_m^{\otimes m} h$  on  $-m(K_X + D) - mf^*\Delta + A_X + D$  is smooth on  $X \setminus f^{-1}(Z_m)$ .

For the  $\mathbb{Q}$ -effective divisor  $D = \sum_{i=1}^t a_i D_i$ , there is a canonical singular hermitian metric  $h_D$  defined on  $D$ , with the local weight

$$\varphi_D = \sum_{i=1}^t a_i \log |g_i|,$$

where  $g_i \in \Gamma(U, \mathcal{O}_U)$  is a holomorphic function locally defining  $D_i$  on some open set  $U \subset X$ . Therefore, the curvature current

$$i\Theta_{h_D} = [D] \geq 0,$$

and thus  $h_D$  is a singular hermitian metric with neat singularities.

Recall that  $Z_D$  is denoted to be the minimal set containing  $Z$ , such that for every  $y \notin Z_D$ , the pair  $(X_y, D|_{X_y})$  is also lc. Here we denote by  $X_y := f^{-1}(y)$ . Since  $(X, D)$  is lc, thus  $Z_D$  is an at most countable union of proper subvarieties of  $Y$ . Indeed, the set

$$Y_m := \{y \notin Z \mid (X_y, (1 - \frac{1}{m})D|_{X_y}) \text{ is klt}\}$$

is an Zariski open set of  $Y$ . Therefore, we have

$$Z_D = \bigcup_{m=1}^{\infty} Y \setminus Y_m.$$

Thus for the singular hermitian metric  $h_m := \tilde{h}_m^{\otimes m} h h_D^{\otimes m-1}$  on  $-mK_X + A_X - mf^*\Delta$ , the multiplier ideal sheaf

$$\mathcal{J}(h_m^{\frac{1}{m}}|_{X_y}) = \mathcal{J}((1 - \frac{1}{m})D|_{X_y}) = \mathcal{O}_{X_y}$$

for any  $y \notin Z_m \cup Y \setminus Y_m$ . Moreover, the curvature current  $i\Theta_{h_m} \geq \omega$ .

By Theorem 2.4 applied with  $L = -mK_X + A_X - mf^*\Delta$  equipped with the hermitian metric  $h_m$ , the restriction is surjective:

$$H^0(X, mK_{X/Y} - mK_X + A_X - mf^*\Delta + f^*A_Y) \rightarrow H^0(X_y, A_{X|X_y})$$

for any  $y \notin Z_m \cup Y \setminus Y_m$ . In other words, the direct image sheaf

$$f_*(mK_{X/Y} - mK_X + A_X - mf^*\Delta + f^*A_Y) = (-K_Y - \Delta)^m \otimes A_Y \otimes f_*(A_X) \quad (4.1)$$

is generated by global sections over  $Y_m \setminus Z_m$ , and by the assumption that  $f_*(A_X)$  is locally free over  $Y \setminus Z$ , we conclude that the top exterior power

$$\wedge^r((-K_Y - \Delta)^m \otimes A_Y \otimes f_*(A_X)) = (-K_Y - \Delta)^{rm} \otimes A_Y^r \otimes \det f_*(A_X) \otimes \mathcal{I}$$

is also generated by global sections over  $Y_m \setminus Z_m$ . In particular, for every  $m \in \mathbb{N}$ , the base locus

$$\text{Bs}((-K_Y - \Delta)^{rm} \otimes A_Y^r \otimes \det f_*(A_X)) \subset Z_m \bigcup Y \setminus Y_m. \quad (4.2)$$

By our choice of  $A_Y$ ,  $rA_Y + \det f_*(A_X)$  is an ample line bundle on  $Y$ , thus let  $m$  tends to infinity, we obtain the pseudo-effectivity of  $-K_Y - \Delta$ . Moreover, from (4.2) we see that the non-nef locus

$$\mathbf{B}_-(-K_Y - \Delta) \subset \bigcup_{m=1}^{\infty} Z_m \bigcup Y \setminus Y_m = f(\mathbf{B}_-(-K_X - D - f^*\Delta)) \bigcup Z_D.$$

Hence Theorem D is proved.  $\square$

If  $f$  is a smooth fibration,  $\text{Supp}(D)$  is a simple normal crossing divisor, and  $\text{Supp}(D)$  is relatively normal crossing over  $Y$ , then the condition that  $(X, D)$  is lc implies that  $(X_y, D|_{X_y})$  is also lc for every  $y \in Y$ . Thus  $Z_D = \emptyset$ . If  $-K_X - D - f^*\Delta$  is nef, then  $\mathbf{B}_-(-K_X - D - f^*\Delta) = \emptyset$ . Thus from Theorem D,  $\mathbf{B}_-(-K_Y - \Delta)$  is also empty which implies that  $-K_Y - \Delta$  is nef. This completes our proof of Theorem 1.3.

By setting  $D = 0$  and  $\Delta = 0$  in Theorem 1.3, the following theorem by Miyaoka is a direct consequence.

**Theorem 4.1.** (Miyaoka) *Let  $f : X \rightarrow Y$  be a smooth morphism between smooth projective manifolds  $X$  and  $Y$ . If  $-K_X$  is nef, then so is  $-K_Y$ .*

**Remark 4.1.** *The original proof of Miyaoka [Miy93] relies on the mod  $p$  reduction arguments. There is also another Hodge theoretic proof by Fujino and Gongyo without using the mod  $p$  reduction arguments [FG14].*

**Remark 4.2.** *In [BBP13], for any pseudo-effective line bundle  $L$ ,  $\mathbf{B}_-(L)$  is called restricted base locus of  $L$ , and the non-nef locus  $\text{NNeft}(L)$  [BBP13, Definition 1.7] is defined in terms of the asymptotic or numerical vanishing orders attached to  $|L|$ . If the underlying projective variety  $X$  is smooth, then we have*

$$\mathbf{B}_-(L) = \text{NNeft}(L).$$

*Since in this paper we always assume that  $X$  and  $Y$  are smooth projective manifolds, we do not distinguish these two equivalent objects.*

**Remark 4.3.** *In [CZ13], M. Chen and Q. Zhang proved the similar result that, for the surjective morphism from the log canonical pair  $(X, D)$  onto a  $\mathbb{Q}$ -Gorenstein variety  $Y$ , if  $-(K_X + D)$  is nef, then  $-K_Y$  is pseudo-effective. In a very recent preprint [Ou17], W. Ou extended the theorem by Chen-Zhang to the rational dominant maps, which was a crucial step in his proof of the generic nefness conjecture for tangent sheaves by T. Peternell [Pet12, Conjecture 1.5].*

## 5 ON THE IMAGES OF WEAK KLT FANO MANIFOLDS

One says that a projective manifold  $X$  is weak Fano if  $-K_X$  is big and nef. In the series of articles [FG12] and [FG14], Fujino and Gongyo studied the image of weak Fano manifolds. They proved the following theorem:

**Theorem 5.1.** (*Fujino-Gongyo*) *Let  $f : X \rightarrow Y$  be a smooth fibration between two smooth manifolds  $X$  and  $Y$ . If  $X$  is weak Fano, then so is  $Y$ .*

In this section, we are going to prove a more general theorem as follows:

**Theorem 5.2.** *Let  $f : X \rightarrow Y$  be a surjective morphism between two smooth manifolds  $X$  and  $Y$ . Let  $D$  be an effective  $\mathbb{Q}$ -divisor such that  $(X, D)$  is klt. Let  $\Delta$  be a (not necessarily effective)  $\mathbb{Q}$ -divisor on  $Y$ . If  $-K_X - D - f^*\Delta$  is big and its non-nef locus  $\mathbf{B}_-(-K_X - D - f^*\Delta)$  does not project onto  $Y$ , then  $-K_Y - \Delta$  is big.*

*Proof.* Take a very ample line bundle  $A_Y$  over  $Y$  such that  $A_Y$  generates  $n + 1$  jets everywhere. Since  $-K_X - D - f^*\Delta$  is big, we can find a positive integer  $a$  such that  $-a(K_X + D + f^*\Delta) - 2f^*A_Y$  is effective. Fix any effective divisor  $E \in |-a(K_X + D + f^*\Delta) - 2f^*A_Y|$ . Since  $(X, D)$  is klt, then there exists a positive integer  $m > a$  such that the multiplier ideal sheaf

$$\mathcal{J}\left(\frac{1}{m-1}E|_{X_y}\right) = \mathcal{J}\left(\frac{m}{m-1}D|_{X_y}\right) = \mathcal{O}_{X_y} \quad (5.1)$$

for the generic fiber  $X_y$ . We can also find a singular hermitian metric  $h_1$  with neat singularities on  $-(m^2 - a)(K_X + D + f^*\Delta)$  such that  $i\Theta_{h_1} \geq \tilde{\omega}$  for some Kähler metric  $\tilde{\omega}$  on  $X$ . Take some small rational number  $\epsilon > 0$  such that  $\mathcal{J}(h_1^\epsilon|_{X_y}) = \mathcal{O}_{X_y}$  for the generic fiber  $X_y$ .

On the other hand, since the non-nef locus  $\mathbf{B}_-(-K_X - D - f^*\Delta)$  does not project onto  $Y$ , from the proof of Theorem D in Section 4, we can find a singular hermitian metric  $h_\epsilon$  over  $-(m^2 - a)(K_X + D + f^*\Delta)$  with neat singularities, such that  $i\Theta_{h_\epsilon} \geq -\epsilon\tilde{\omega}$  and the singularities of  $h_\epsilon$  does not project onto  $Y$ . Set  $h := h_1^\epsilon h_\epsilon^{1-\epsilon}$  which is also a hermitian metric on  $-(m^2 - a)(K_X + D + f^*\Delta)$ , then we have  $i\Theta_h \geq \epsilon^2\tilde{\omega}$  and the multiplier ideal sheaf

$$\mathcal{J}(h|_{X_y}) = \mathcal{O}_{X_y} \quad (5.2)$$

for the generic fiber  $X_y$ .

Take a generic fiber  $X_y$  of  $f$  such that  $y$  is not the regular value of  $f$ , and both (5.1) and (5.2) are satisfied. We equip the line bundle  $-m^2(K_X + D + f^*\Delta) - 2f^*A_Y + m^2D$  with the singular hermitian metric  $h_0 := h_E h h_D^{\otimes m^2}$ , where  $h_E$  (resp.  $h_D$ ) is the tautological singular hermitian metric on  $-a(K_X + D + f^*\Delta) - 2f^*A_Y$  (resp.  $D$ ) induced by the effective divisor  $E$  (resp.  $D$ ), such that

$$i\Theta_{h_E} = [E] \text{ (resp. } i\Theta_{h_D} = [D]).$$

Then we claim that the multiplier ideal sheaf  $\mathcal{J}(h_0^{\frac{1}{m^2}}|_{X_y}) = \mathcal{O}_{X_y}$ . Indeed, for any  $s \in \mathcal{O}_{X_y, z}$ , let  $\varphi_E, \varphi_D$  and  $\varphi$  be the weights of the metric  $h_E, h_D$  and  $h$  on a small neighborhood  $U \subset X_y$  of a point  $z \in X_y$ . Then by the Hölder inequality we have

$$\int_U |s|^2 e^{-\frac{\varphi_E + \varphi}{m^2} + \varphi_D} \leq \left( \int_U |s|^2 e^{-\varphi} \right)^{\frac{1}{m^2}} \cdot \left( \int_U |s|^2 e^{-\frac{\varphi_E}{m-1}} \right)^{\frac{m-1}{m^2}} \cdot \left( \int_U |s|^2 e^{-\frac{m}{m-1}\varphi_D} \right)^{\frac{m-1}{m}} < +\infty.$$

Here we use the conditions (5.1) and (5.2). By applying Theorem 2.4 with  $L = -m^2(K_X + f^*\Delta) - 2f^*A_Y$  endowed with the singular hermitian metric  $h_0$ , we obtained the desired surjectivity:

$$H^0(X, m^2 K_{X/Y} + (-m^2 K_X - m^2 \Delta - 2f^*A_Y) + f^*A_Y) \rightarrow H^0(X_y, f^*(-m^2 K_Y - m^2 \Delta - A_Y)|_{X_y}) = \mathbb{C}.$$

Therefore,  $-m^2 K_Y - m^2 \Delta - A_Y$  is an effective line bundle, which also shows that  $-K_Y - \Delta$  is big.  $\square$

Therefore, we can extend Theorem 5.1 to the weak klt Fano cases:

*Proof of Theorem F.* Since  $f$  is a smooth fibration,  $(X, D)$  is klt, and  $(X_y, D|_{X_y})$  is also klt for every  $y \in Y$ , from the very definition of  $Z_D$  in Theorem D we see that  $Z_D = \emptyset$ . By the nefness of  $-(K_X + D) - f^*\Delta$ , the set

$$\mathbf{B}_-(-(K_X + D) - f^*\Delta) = \emptyset.$$

Thus from Theorem D we conclude that  $-K_Y - \Delta$  is nef. The bigness of  $-K_Y - \Delta$  follows from Theorem F directly. This completes the proof.  $\square$

By setting  $D = 0$  and  $\Delta = 0$  in Theorem F, we obtain Theorem 5.1 directly.

**Remark 5.1.** *If we only assume that  $-K_X$  is big, then the following example given in [FG12] shows that, even if  $f$  is smooth,  $-K_Y$  is not big.*

**Example 5.1.** *Let  $E \subset \mathbb{P}^2$  be a smooth cubic curve. Consider  $f : X = \mathbb{P}_E(\mathcal{O}_E \oplus \mathcal{O}_E(1)) \rightarrow E = Y$ . Then, we see that  $-K_X$  is big. However,  $-K_Y$  is not big since  $E$  is a smooth elliptic curve.*

It is noticeable that, in [Pau12] S. Boucksom pointed out that the following theorem, which is a special case of Theorem 1.2 in [Ber09], implies [FG12, Theorem 4.1] or [KMM92, Corollary 2.9]:

**Theorem 5.3.** *(Boucksom-Păun) Let  $f : X \rightarrow Y$  be a smooth fibration between two smooth manifolds. If  $-K_X$  is semi-positive (strictly positive), then  $-K_Y$  is semi-positive (strictly positive).*

Finally, let us mention that, in [FG12], the authors raised the following conjecture, which was solved very recently by C. Birkar and Y. Chen [BC16]:

**Theorem 5.4.** *(Fujino-Gongyo, Birkar-Chen) Let  $f : X \rightarrow Y$  be a smooth fibration between two smooth projective manifolds. If  $-K_X$  is semi-ample, then so is  $-K_Y$ .*

The proof in [BC16] relies on very deep consequences of the minimal model program in birational geometry and of Hodge theory. It is an interesting question to know whether we can use pure analytic methods to give a new proof of this theorem.

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## References

- [AS95] U. Angehrn, Y.-T. Siu: *Effective freeness and point separation for adjoint bundles*. Inventiones mathematicae, 1995, 122(1): 291-308.
- [Ber09] B. Berndtsson: *Curvature of vector bundles associated to holomorphic fibrations*. Annals of mathematics, 2009: 531-560.
- [BBP13] S. Boucksom, A. Broustet, G. Pacienza: *Uniruledness of stable base loci of adjoint linear systems via Mori theory*. Mathematische Zeitschrift, 2013, 275(1-2): 499-507.
- [BC16] C. Birkar, Y. Chen: *Images of manifolds with semi-ample anti-canonical divisor*. Journal of Algebraic Geometry, 2016.
- [BDPP13] S. Boucksom, J.-P. Demailly, M. Paun, T. Peternell: *The pseudo-effective cone of a compact Kähler manifold and varieties of negative Kodaira dimension*. Journal of Algebraic Geometry, 2013, 22(2): 201-248.
- [Bou02] S. Boucksom: *Cônes positifs des variétés complexes compactes*. Thesis, Grenoble 2002
- [BP10] B. Berndtsson, M. Păun: *Bergman kernels and subadjunction* arXiv: 1002.4145v1 [math.AG].



- [Cao16] J. Cao: *Albanese maps of projective manifolds with nef anticanonical bundles*, arXiv:1612.05921 [math.AG].
- [CZ13] M. Chen, Q. Zhang: *On a question of Demailly-Peternell-Schneider*. Journal of the European Mathematical Society 015.5 (2013): 1853-1858.
- [Dem92] J.-P. Demailly: *Singular Hermitian metrics on positive line bundles*. Complex algebraic varieties. Springer Berlin Heidelberg, 1992: 87-104.
- [Dem92b] J.-P. Demailly: *Regularization of closed positive currents and intersection theory*. Journal of Algebraic Geometry, 1992, 1(3): 361-409.
- [Dem96] J.-P. Demailly: *Effective bounds for very ample line bundles*. Inventiones mathematicae, 1996, 124(1-3): 243-261.
- [Dem12] J.-P. Demailly: *Analytic methods in algebraic geometry*. International Press, 2012.
- [Dem15] J.-P. Demailly: *Extension of holomorphic functions defined on non reduced analytic subvarieties*. arXiv:1510.05230, 2015.
- [DPS94] J.-P. Demailly, T. Peternell, M. Schneider: *Compact complex manifolds with numerically effective tangent bundles*. J. Algebraic Geom. vol 3, (1994) number 2, pp. 295-345.
- [DPS01] J.-P. Demailly, T. Peternell, M. Schneider: *Pseudo-effective line bundles on compact Kähler manifolds*. International Journal of Mathematics, 2001, 12(06): 689-741.
- [Dut17] Y. Dutta: *On the effective freeness of the direct images of pluricanonical bundles*. arXiv:1701.08830 [math.AG].
- [EL93] L. Ein, R. Lazarsfeld: *Seshadri constants on smooth surfaces*. Journées de Géométrie Algébrique d'Orsay (Orsay 1992), Astérisque, no. 218, 1993, pp. 177-186.
- [EKL95] L. Ein, O. Küchle, R. Lazarsfeld: *Local positivity of ample line bundles*. Journal of Differential Geometry. Volume 42, Number 2 (1995), 193-219.
- [FG12] O. Fujino, Y. Gongyo: *On images of weak Fano manifolds*. Mathematische Zeitschrift, 2012, 270(1-2): 531-544.
- [FG14] O. Fujino, Y. Gongyo: *On images of weak Fano manifolds II*. Algebraic and Complex Geometry. Springer International Publishing, 2014: 201-207.
- [KMM92] J. Kollár, Y. Miyaoka, S. Mori: *Rational connectedness and boundedness of Fano manifolds*. Journal of Differential Geometry, 1992, 36(3): 765-779.
- [Laz04] R. Lazarsfeld: *Positivity in algebraic geometry I, II*. Springer-Verlag. 2004.
- [Miy93] Y. Miyaoka: *Relative deformations of morphisms and applications to fibre spaces*. Rikkyo Daigaku sugaku zasshi, 1993, 42(1): 1-7.
- [Ou17] W. Ou: *On generic nefness of tangent sheaves*. arXiv:1703.03175 [math.AG].
- [Pau07] M. Păun: *Siu's invariance of plurigeners: a one-tower proof*. Journal of Differential Geometry, 2007, 76(3): 485-493.
- [Pau12] M. Păun. *Relative adjoint transcendental classes and Albanese maps of compact Kähler manifolds with nef Ricci curvature*. arXiv: 1209.2195 [math.AG].
- [Pet12] T. Peternell: *Varieties with generically nef tangent bundles*. Journal of the European Mathematical Society, 2012, 14(2): 571-603.
- [PS14] M. Popa, C. Schnell: *On direct images of pluricanonical bundles*. Algebra & Number Theory, 2014, 8(9): 2273-2295.

- [PT14] M. Păun, S. Takayama: *Positivity of twisted relative pluricanonical bundles and their direct images*. arxiv 1409.5504 [math.AG].
- [Siu97] Y.-T. Siu: *Invariance of plurigenera*. Inventiones mathematicae, 1998, 134(3): 661-673.